Branching processes in random environment which extinct at a given moment*

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Abstract

Let $\{Z_n, n \geq 0\}$ be a critical branching process in random environment and let T be its moment of extinction. Under the annealed approach we prove, as $n \to \infty$, a limit theorem for the number of particles in the process at moment n given $T = n + 1$ and a functional limit theorem for the properly scaled process $\{Z_{nt}, \delta \leq t \leq 1 - \delta\}$ given $T = n + 1$ and $\delta \in (0, 1/2)$.

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1 Introduction and main results

The model of branching processes in random environment which we are dealing with in this paper was introduced by Smith and Wilkinson [4]. To give a formal definition of these processes denote $\mathcal{M}$ the space of probability measures on $\mathbb{N}_0 := \{0, 1, 2, ...\}$ and let Q be a random variable taking values in $\mathcal{M}$. An infinite sequence $\Pi = (Q_1, Q_2, \ldots)$ of i.i.d. copies of Q is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is called a *branching process in the random environment* Π, if Z_0 is independent of Π and, given Π, the process $Z = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\left(Z_n \mid Z_{n-1} = z_{n-1}, \Pi = (q_1, q_2, \ldots)\right) = \mathcal{L}\left(\xi_{n1} + \cdots + \xi_{nz_{n-1}}\right) \quad (1)$$

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for every $n \geq 1$, $z_{n-1} \in \mathbb{N}_0$ and $q_1, q_2, \ldots \in \mathcal{M}$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution q_n. We can write this as

$$Z_n := \sum_{i=1}^{Z_{n-1}} \xi_{ni}, \tag{2}$$

where, given the environment, Z is an ordinary inhomogeneous Galton-Watson process. Thus, Z_n is the nth generation size of the population and Q_n is the distribution of the number of children of an individual at generation $n-1$. We will denote the corresponding probability measure on the underlying probability space by $\mathbf{P}$.

In what follows we identify Q and $Q_n, n = 1, 2, ...,$ with (random) generating functions

$$f(s) \ := \ \sum_{i=0}^{\infty} s^i \, Q(\{i\}) =: \mathbf{E}\left[s^{\xi} | Q\right], \qquad 0 \leq s \leq 1 \ ,$$

and

$$f_n(s) \ := \ \sum_{i=0}^{\infty} s^i \, Q_n(\{i\}) =: \mathbf{E}\left[s^{\xi_n} | Q_n\right], \qquad 0 \leq s \leq 1 \ ,$$

and make no difference between the tuples $\Pi = (Q_1, Q_2, \ldots)$ and $\mathbf{f} = (f_1, f_2, \ldots)$.

Let

$$X_k := \ln f_k'(1), \qquad \eta_k := \frac{f_k''(1)}{2\left(f_k'(1)\right)^2},$$

and

$$(f, X, \eta) \stackrel{d}{=} (f_1, X_1, \eta_1).$$

The sequence

$$S_0 := 0, \ S_n := X_1 + ... + X_n, \ n \in \mathbb{N} := \{1, 2, ...\}$$

is called the *associated random walk* of the corresponding branching process in random environment (BPRE).

Following [1] we call a BPRE critical if $\limsup_{n\to\infty} S_n = +\infty$ and $\liminf_{n\to\infty} S_n = -\infty$ both with probability 1.

Let

$$T = \min\{k \geq 0 : Z(k) = 0\}$$

be the extinction moment of the critical BPRE. The aim of the paper is to study, as $n \to \infty$, the behavior of the process $\{Z(m), 1 \leq m \leq n\}$ given $T = n+1$. Critical BPRE's conditioned on extinction at a given moment were investigated in [3] and [8] under the annealed approach and in [7] under the quenched approach. In all the papers it is assumed that the functions $f_n(s)$ are fractional-linear, namely,

$$\frac{1}{1 - f_n(s)} = \frac{e^{-X_n}}{1-s} + \eta_n.$$

In [8] the asymptotic behavior of the probability $\mathbf{P}(T=n)$ as $n \to \infty$ is found and a conditional functional limit theorem for the properly scaled process $\{Z(m), 1 \le m \le n\}$ given $T = n+1$ is proved under the assumption that the distribution of X belongs to the domain of attraction of a stable law with parameter $\alpha \in (0,2)$. It was shown that in this case the phenomena of sudden extinction of the process takes place. Namely, if the process survives for a long time ($T = n+1 \to \infty$) then $\log Z_{[nt]}$ grows, roughly speaking, as $n^{1/\alpha}$ up to moment n and then the process instantly dies out. In particular, $\log Z_n$ is of order $n^{1/\alpha}$. This may be interpreted as the evolution of the process in a favorable environment up to moment n and sudden extinction of the population at moment $T = n+1 \to \infty$ because of a very unfavorable, even "catastrophic" environment at moment n.

For the case $\mathbf{E}X^2 < \infty$ the asymptotic behavior of the probability $\mathbf{P}(T = n+1)$ as $n \to \infty$ was investigated in [3]. However, no functional limit theorem was proved. We fill this gap in the present paper and establish a conditional functional limit theorem for the process

$$\left\{ Z_{nt} e^{-S_{nt}}, \delta \le t \le 1-\delta \,\middle|\, T = n+1 \right\}, \, \delta \in (0, 1/2),$$

as $n \to \infty$ and, in addition, show that the conditional law $\mathcal{L}(Z_n \,|\, T = n+1)$ weakly converges to a law concentrated on natural numbers. Thus, contrary to the case considered in [8], the phenomenon of sudden extinction is absent if $\mathbf{E}X^2 < \infty$ under the annealed approach.

Note that paper [7] demonstrates that in case of the quenched approach the phenomenon of sudden extinction does not occur if X belongs to the domain of attraction of a stable law with parameter $\alpha \in (0,2]$.

Now we list the basic conditions imposed in this paper on the characteristics of our BPRE.

Assumption A1. There exists a constant $\chi \in (0, 1/2)$ such that

$$0 < \chi \le f(0) \le 1 - \chi, \quad \eta \ge \chi \tag{3}$$

with probability 1.

Assumption A2. The distribution of X has zero mean, finite and positive variance σ^2 and is non-lattice.

Let

$$\zeta(a) := \sum_{y=a}^{\infty} y^2 Q(\{y\}) / m(Q)^2, \, a \in \mathbb{N} := \{1, 2, ...\}.$$

Assumption A3. For some $\varepsilon > 0$ and some $a \in \mathbb{N}$

$$\mathbf{E}\left[\left(\log^+ \zeta(a) \right)^{2+\varepsilon} \right] < \infty,$$

where $\log^+ x := \log(\max\{x, 1\})$.

Here are our main results.

Theorem 1 *Under A1 to A3, as* $n \to \infty$,

$$\mathbf{P}(T = n+1) \sim cn^{-3/2},$$

where $c \in (0, \infty)$, *and*

$$\mathcal{L}(Z_n \,|\, T = n+1) \to \mathcal{L}(Y)$$

weakly, where Y *is a non-degenerate random variable finite with probability 1.*

Theorem 2 *Under A1 to A3 for any* $\delta \in (0, 1/2)$, *as* $n \to \infty$,

$$\mathcal{L}\left(\frac{Z_{nt}}{e^{S_{nt}}}, t \in [\delta, 1-\delta] \,\Big|\, T = n+1\right) \Rightarrow \mathcal{L}(W_t, t \in [\delta, 1-\delta]),$$

where the limiting process W_t *has a.s. constant trajectories, i.e.,*

$$\mathbf{P}(W_t = W \text{ for all } t \in (0,1)) = 1,$$

and

$$\mathbf{P}(0 < W < \infty) = 1.$$

Here the symbol $\Rightarrow$ means weak convergence with respect to the Skorokhod topology in the space $D[\delta, 1-\delta]$ of cadlag functions on the interval $[\delta, 1-\delta]$.

The proofs of the above results are based on the approach initiated in [1] and developed recently in [2] and use the fact that the asymptotic behavior of the critical BPRE's is, essentially, specified by the properties of its associated random walk.

2 Some auxiliary results

In this section we give a list of general results related with an oscillating random walk $S_0, S_k = S_0 + X_1 + ... + X_k$ with no referring to the critical BPRE's and we allow here S_0 to be a random variable for technical reason. These results are basically taken from [2] and are established under the following assumption.

Assumption A4. There are numbers $c_n \to \infty$ such that the sequence S_n/c_n converges in distribution to an $\alpha-$ stable law which is neither concentrated on $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_- := (-\infty, 0]$. It is nonlattice.

Introduce the random variables

$$M_n := \max(S_1, ..., S_n), \quad L_n := \min(S_1, ..., S_n)$$

and, given $S_0 = 0$, the right-continuous functions $u : \mathbb{R} \to \mathbb{R}_+$ and $v : \mathbb{R} \to \mathbb{R}_+$ specified by the equalities

$$u(x) := 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \le x, M_k < 0), \quad x \in \mathbb{R}_+,$$

$$v(x) := 1 + \sum_{k=1}^{\infty} \mathbf{P}\left(-S_k > x, L_k \geq 0\right), \quad x \in \mathbb{R}_-,$$

with $u(0) = v(0) = 1$ and $u(-x) = v(x) = 0, \quad x \in \mathbb{R}_+$.

One may check (see, for instance, [1] and [2]) that for any oscillating random walk

$$\mathbf{E}\left[u(x+X); X + x \geq 0\right] = u(x), \quad x \in \mathbb{R}_+, \tag{4}$$

$$\mathbf{E}\left[v(x+X); X + x < 0\right] = v(x), \quad x \in \mathbb{R}_-. \tag{5}$$

By u and v we construct two probability measures $\mathbf{P}^+$ and $\mathbf{P}^-$. To this aim let $O_1, O_2, ...$ be a sequence of identically distributed random variables in a state space $\mathcal{D}$, adapted to a filtration $(\mathcal{F}_n, n \in \mathbb{N}_0)$ (possibly larger than the filtration generated by $(O_n, n \geq 1)$) such that for all n, O_{n+1} is independent of $\mathcal{F}$ and, in particular, $(O_n, n \geq 1)$ is a sequence of i.i.d. random variables. Let, further, $R_0, R_1, ...$ be a sequence of random variables in a state space $\mathcal{S}$ and also adapted to $\mathcal{F}$. We assume that the increments $(X_n, n \geq 1)$ of the random walk S are such that for all n, X_n are measurable with respect to the σ-field generated by O_n and S_0 is $\mathcal{F}_0$-measurable.

Now for any bounded and measurable function $g : \mathcal{S} \to \mathbb{R}$, we construct probability measures $\mathbf{P}_x^+, x \geq 0$, and $\mathbf{P}_x^-, x \leq 0$, fulfilling for each n the equalities

$$\mathbf{E}_x^+\left[g\left(R_0, ..., R_n\right)\right] = \frac{1}{u(x)} \mathbf{E}_x\left[g\left(R_0, ..., R_n\right) u(S_n); L_n \geq 0\right]$$

and

$$\mathbf{E}_x^-\left[g\left(R_0, ..., R_n\right)\right] = \frac{1}{v(x)} \mathbf{E}_x\left[g\left(R_0, ..., R_n\right) v(S_n); M_n < 0\right].$$

Using (4)-(5) it is not difficult to check (see [1] and [2] for more detail) that the measures specified in this way are consistent in n.

Let $d_n = (nc_n)^{-1}$. In the sequel if no otherwise is stated, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$ and $a_n \to a$ if $\lim_{n\to\infty} a_n = a$.

Let

$$\tau(n) = \min\left\{i \leq n : S_i = \min\left(S_0, S_1, ..., S_n\right)\right\}$$

be the moment of the first random walk minimum up to time n.

The next three results are borrowed from [2].

Lemma 3 *([2], Proposition 2.1) Under assumption **A4** for $x \geq 0$, $\theta > 0$*

$$\mathbf{E}_x\left[e^{-\theta S_n}; L_n \geq 0\right] \sim s(0) d_n u(x) \int_0^{\infty} e^{-\theta z} v(-z) dz$$

and for $x \leq 0$

$$\mathbf{E}_x\left[e^{\theta S_n}; \tau(n) = n\right] \sim s(0) d_n v(x) \int_0^{\infty} e^{-\theta z} u(z) dz.$$

In particular, if $\sigma^2 := \mathbf{E}X^2 < \infty$ then, for some positive constants K_1 and K_2

$$\mathbf{E}\left[e^{-S_n}; L_n \geq 0\right] \sim K_1 n^{-3/2} \text{ and } \mathbf{E}\left[e^{S_n}; \tau(n) = n\right] \sim K_2 n^{-3/2}. \tag{6}$$

For $\theta > 0$, let μ_θ, ν_θ be the probability measures on $\mathbb{R}_+$ and $\mathbb{R}_-$ given by their densities

$$\mu_\theta(dz) := c_{1\theta} e^{-\theta z} u(z) I(z \geq 0)\, dz, \quad \nu_\theta(dz) := c_{2\theta} e^{\theta z} \nu(z) I(z < 0)\, dz, \tag{7}$$

where

$$c_{1\theta}^{-1} = \int_0^\infty e^{-\theta z} u(z) dz, \quad c_{2\theta}^{-1} = \int_0^\infty e^{-\theta z} v(-z) dz. \tag{8}$$

Lemma 4 *([2], Proposition 2.7) Let $0 < \delta < 1$. Let $U_n = g_n\left(R_0, ..., R_{[\delta n]}\right)$, $n \geq 1$, be random variables with values in an Euclidean (or polish) space S such that, as $n \to \infty$*

$$U_n \to U_\infty \quad \mathbf{P}^+ \ a.s.$$

for some S-valued random variable U_∞. Also, let $V_n = h_n\left(Q_1, ..., Q_{[\delta n]}\right)$, $n \geq 1$, be random variables with values in an Euclidean (or polish) space S' such that

$$V_n \to V_\infty \quad \mathbf{P}_x^- \ a.s.$$

for all $x \leq 0$ and some S'-valued random variable V_∞. Denote

$$\tilde{V}_n := h_n\left(Q_n, ..., Q_{n-[\delta n]+1}\right).$$

*Under assumption **A4** for $\theta > 0$ and any bounded continuous function $\varphi : S \times S' \times \mathbb{R} \to \mathbb{R}$ as $n \to \infty$*

$$\frac{\mathbf{E}\left[\varphi\left(U_n, \tilde{V}_n, S_n\right) e^{-\theta S_n}; L_n \geq 0\right]}{\mathbf{E}\left[e^{-\theta S_n}; L_n \geq 0\right]} \to \int \cdots \int \varphi(u, v, -z) \quad \mathbf{P}^+(U_\infty \in du)\, \mathbf{P}_z^-(V_\infty \in dv)\, \nu_\theta(dz).$$

The dual version of Lemma 4 looks as follows.

Lemma 5 *([2], Proposition 2.9) Let $U_n, V_n, \tilde{V}_n, n = 1, 2, ..., \infty$, be as in Lemma 4 and now fulfilling, as $n \to \infty$*

$$U_n \to U_\infty \quad \mathbf{P}_x^+ - \ a.s., \quad V_n \to V_\infty \quad \mathbf{P}^- - \ a.s.$$

*for all $x \geq 0$. Under assumption **A4** for any bounded continuous function $\varphi : S \times S' \times \mathbb{R} \to \mathbb{R}$ and for $\theta > 0$ as $n \to \infty$*

$$\frac{\mathbf{E}\left[\varphi\left(U_n, \tilde{V}_n, S_n\right) e^{\theta S_n}; \tau(n) = n\right]}{\mathbf{E}\left[e^{\theta S_n}; \tau(n) = n\right]} \to \int ... \int \varphi(u, v, -z)\, \mathbf{P}_z^+(U_\infty \in du)\, \mathbf{P}^-(V_\infty \in dv)\, \mu_\theta(dz).$$

Remark 6 *It is easy to see (by introducing formal arguments) that the statements of the lemmas are valid for any integer-valued function $w(n)$ such that $w(n) \leq \delta n$ for all sufficiently large n. Then the functions g_n and h_n can be viewed as functions also of the missing variables). Later on we use this fact with no additional reference.*

3 Discrete limit distribution

Introduce the compositions

$$f_{k,n}(s) \ := \ f_{k+1}(f_{k+2}(\cdots f_n(s)\cdots)) \ , \ 0\le k<n, \quad f_{n,n}(s) \ := s, \qquad (9)$$

and

$$f_{k,0}(s) \ := \ f_k(f_{k-1}(\cdots f_1(s)\cdots)), \ k>0.$$

In this notation we may rewrite the distributional identity (1) for $k\le n$ as

$$\mathbf{E}[s^{Z_n} \mid \mathbf{f}, Z_k] = \mathbf{E}[s^{Z_n} \mid \Pi, Z_k] \ = \ f_{k,n}(s)^{Z_k} \quad \mathbf{P}\text{–a.s.} \qquad (10)$$

For $0\le k\le n$ and $S_0:=0$ let

$$a_{k,n} := e^{-(S_n-S_k)}, \quad a_n := a_{0,n} = e^{-S_n},$$

$$b_{k,n} := \sum_{i=k}^{n-1}\eta_{i+1}e^{-(S_i-S_k)}, \quad b_n := b_{0,n} = \sum_{i=0}^{n-1}\eta_{i+1}e^{-S_i}. \qquad (11)$$

Lemma 7 *(see, for instance, [5]) In the fractional-linear case for any* $0\le j<n$

$$(1-f_{j,n}(s))^{-1} = \frac{a_{j,n}}{1-s} + b_{j,n}. \qquad (12)$$

In particular,

$$(1-f_{j,n}(0))^{-1} = a_{j,n} + b_{j,n} \qquad (13)$$

and

$$\begin{aligned}(1-f_{0,n}(s))^{-1} &= \frac{a_n}{1-s} + b_n = (1-f_{0,j}(f_{j,n}(s)))^{-1} \\ &= \frac{a_j}{1-f_{j,n}(s)} + b_j = \frac{a_j a_{j,n}}{1-s} + b_j + a_j b_{j,n}. \qquad (14)\end{aligned}$$

Lemma 8 *(see Lemma 3.1 in [2]) If conditions A3 - A4 are valid then for any* $x\ge 0$

$$B^+ := \lim_{n\to\infty} b_n = \sum_{i=0}^{\infty}\eta_{i+1}e^{-S_i} < \infty \quad \mathbf{P}_x^+\text{- } a.s.$$

and for any $x\le 0$

$$B^- := \lim_{n\to\infty}\sum_{i=1}^{n}\eta_i e^{S_i} = \sum_{i=1}^{\infty}\eta_i e^{S_i} < \infty \quad \mathbf{P}_x^-\text{- } a.s.$$

Let $\kappa : \mathbb{R} \to \mathbb{R}_+$ be the function specified by the equality

$$\kappa(y) := yI(y > 0),$$

where $I(A)$ is the indicator of the event A, and let, for positive constants α, β, γ

$$\phi(\alpha, \beta, \gamma; u, v, x) := \frac{1}{e^{-x}\alpha + \beta + \gamma(\kappa(u) + e^{-x}\kappa(v))}, \tag{15}$$

and

$$\Phi(u, v, x) = \Phi(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2; u, v, x) := \prod_{i=1}^{2} \phi(\alpha_i, \beta_i, \gamma_i; u, v, x).$$

Clearly, $\Phi(u, v, x)$ is continuous in $\mathbb{R}^3$ and bounded by $\beta_1^{-1}\beta_2^{-1}$.

Lemma 9 *Under the conditions A3-A4, for any $\alpha_i > 0, \beta_i > 0, \gamma_i > 0, i = 1, 2$*

$$\mathbf{E}\left[\frac{e^{-S_n}}{\prod_{i=1}^{2}(e^{-S_n}\alpha_i + \beta_i + \gamma_i b_n)}; L_n \geq 0\right] / \mathbf{E}\left[e^{-S_n}; L_n \geq 0\right]$$
$$\to \iiint \Phi(u, v, -z)\, \mathbf{P}^+\left(B^+ \in du\right) \mathbf{P}_z^-\left(B^- \in dv\right) \nu_1(dz).$$

Proof. We have

$$\begin{aligned} e^{-S_n}\alpha + \beta + \gamma b_n &= e^{-S_n}\alpha + \beta + \gamma(U_{[n/2]} + e^{-S_n}\tilde{V}_{[n/2]}) \\ &= 1/\phi\left(\alpha, \beta, \gamma; U_{[n/2]}, \tilde{V}_{[n/2]}, S_n\right), \end{aligned}$$

where

$$U_{[n/2]} = g_{[n/2]}\left(Q_1, ..., Q_{[n/2]}\right) := b_{[n/2]}$$

with b_n specified by (11), and

$$\tilde{V}_{[n/2]} = h_{[n/2]}\left(Q_n, ..., Q_{n-[n/2]+1}\right) := \sum_{i=0}^{n-[n/2]} \eta_{i+[n/2]+1} e^{S_n - S_{i+[n/2]}}.$$

By Lemma 8 as $n \to \infty$ for any $x \geq 0$

$$U_{[n/2]} = b_{[n/2]} \to B^+ \quad \mathbf{P}_x^+\text{-a.s.} \tag{16}$$

and, for any $x \leq 0$

$$V_{[n/2]} = \sum_{i=1}^{[n/2]+1} \eta_i e^{S_i} \to B^- \quad \mathbf{P}_x^-\text{-a.s.} \tag{17}$$

Applying Lemma 4 to $\Phi\left(U_n, \tilde{V}_n, S_n\right)$ completes the proof of the desired statement.

Let, for $\alpha > 0$

$$\psi(\alpha; u, v, x) := \frac{1}{\alpha + e^{x}\kappa(u) + \kappa(v)} \tag{18}$$

and

$$\Psi(u, v, x) = \Psi(\alpha_1, \alpha_2; u, v, x) := \prod_{i=1}^{2} \psi(\alpha_i; u, v, x). \tag{19}$$

Lemma 10 *Under the conditions A3-A4, for any* $\alpha_1 > 0, \alpha_2 > 0$

$$\mathbf{E}\left[\frac{e^{S_n}}{\prod_{i=1}^{2}(\alpha_i + e^{S_n}b_n)}; \tau(n) = n\right] / \mathbf{E}\left[e^{S_n}; \tau(n) = n\right]$$
$$\to \iiint \Psi(u, v, -z)\,\mathbf{P}_z^{+}\left(B^{+} \in du\right)\mathbf{P}^{-}\left(B^{-} \in dv\right)\mu_1(dz).$$

Proof. We have

$$\alpha + e^{S_n}b_n = \alpha + e^{S_n}U_{[n/2]} + \tilde{V}_{[n/2]} = 1/\psi\left(\alpha; U_{[n/2]}, \tilde{V}_{[n/2]}, S_n\right),$$

where $U_{[n/2]}$ and $\tilde{V}_{[n/2]}$ are the same as in Lemma 9. Now using (16) and (17) once again it is not difficult to complete the proof of the lemma.

Proof of Theorem 1. For $s \in [0, 1)$ denote

$$X_f(s) := \frac{sf(0)}{1 - sf(0)}, \quad G_n(s) := 1 - f_{0,n}(s) = \left(\frac{a_n}{1-s} + b_n\right)^{-1}$$

and let

$$\begin{aligned}
\Delta_n(s) := \quad & f_{0,n}(sf(0)) - f_{0,n}(0) \\
= \quad & (1 - f_{0,n}(0))(1 - f_{0,n}(sf(0)))e^{-S_n}\frac{sf(0)}{1 - sf(0)} \qquad (20) \\
= \quad & G_n(f(0))G_n(sf(0))X_f(s)e^{-S_n} \\
= \quad & (a_n + b_n)^{-1}\left(\frac{a_n}{1 - sf(0)} + b_n\right)^{-1} X_f(s)e^{-S_n},
\end{aligned}$$

where we have used the explicit form of $f_{0,n}(s)$ and the equality $f \overset{d}{=} f_{n+1}$. It is not difficult to check that

$$\begin{aligned}
\mathbf{E}\left[s^{Z_n}; T = n+1\right] &= \mathbf{E}\left[s^{Z_n}; Z_n > 0, Z_{n+1} = 0\right] \\
&= \mathbf{E}\left[(sf_{n+1}(0))^{Z_n} I(Z_n > 0)\right] \\
&= \mathbf{E}\left[(sf_{n+1}(0))^{Z_n} - I(Z_n = 0)\right] = \mathbf{E}\Delta_n(s) \\
&= D_1(N, n) + D_2(N, n) + D_3(N, n),
\end{aligned}$$

where

$$D_1(N, n) := \sum_{j=0}^{N} \mathbf{E}\left[\Delta_n(s); \tau(n) = j\right],$$

$$D_2(N,n) := \sum_{j=N+1}^{n-N-1} \mathbf{E}\left[\Delta_n\left(s\right); \tau(n)=j\right],$$

$$D_3(N,n) := \sum_{j=n-N}^{n} \mathbf{E}\left[\Delta_n\left(s\right); \tau(n)=j\right].$$

By (20), the evident inequalities

$$1-f_{0,n}\left(0\right)=\min_{1\leq k\leq n}\left(1-f_{0,k}\left(0\right)\right)\leq \min_{1\leq k\leq n} e^{S_k}=e^{S_{\tau(n)}},$$

Assumption A1 and the estimates

$$X_f(s)\leq X_f(1)\leq(1-\chi)\chi^{-1}=:\rho \tag{21}$$

following from (3) we obtain

$$\Delta_n\left(s\right)\leq\left(1-f_{0,n}\left(0\right)\right)^2 e^{-S_n}X_f(1)\leq\rho\chi e^{2S_{\tau(n)}-S_n}\leq\rho e^{2S_{\tau(n)}-S_n}.$$

Using this estimate, the asymptotic relation (6) and the duality principle for random walks it is not difficult to show that for any $\varepsilon>0$ one can find $N=N(\varepsilon)$ such that for all sufficiently large $n\geq 2N+1$

$$\begin{aligned} D_2(N,n) &\leq \rho\sum_{j=N+1}^{n-N-1}\mathbf{E}\left[e^{2S_{\tau(n)}-S_n};\tau(n)=j\right] \\ &= \rho\sum_{j=N+1}^{n-N-1}\mathbf{E}\left[e^{S_j};\tau(j)=j\right]\mathbf{E}\left[e^{-S_{n-j}};L_{n-j}\geq 0\right] \\ &\leq const\times\sum_{j=N+1}^{n-N-1}\frac{1}{j^{3/2}}\frac{1}{(n-j)^{3/2}}\leq\varepsilon n^{-3/2}. \end{aligned} \tag{22}$$

Further, for fixed j let

$$\tilde{V}_{[(n-j)/2]} := \sum_{i=0}^{n-[(n-j)/2]} \eta_{i+[(n-j)/2]+1}e^{S_n-S_{i+[(n-j)/2]}}.$$

By (14) and (15) we have for $s\in[0,1)$

$$\begin{aligned} G_n(sf(0)) &= \left(\frac{a_j a_{j,n}}{1-sf(0)}+b_j+a_j b_{j,n}\right)^{-1} \\ &= \left(\frac{a_j}{1-sf(0)}e^{-(S_n-S_j)}+b_j+a_j\left(b_{j,[(n-j)/2]}+e^{-(S_n-S_j)}\tilde{V}_{[(n-j)/2]}\right)\right)^{-1} \\ &= \phi\left(\frac{a_j}{1-sf(0)},b_j,a_j;b_{j,[(n-j)/2]},\tilde{V}_{[(n-j)/2]},S_n-S_j\right)\leq 1. \end{aligned} \tag{23}$$

Denote by $\mathcal{F}_j^*$ the σ-algebra generated by $Q, Q_1, .., Q_j$ and introduce a temporary notation

$$\alpha_1 = a_j, \quad \alpha_2 = \frac{a_j}{1 - sf(0)}.$$

Then

$$\mathbf{E}\left[\Delta_n(s); \tau(n) = j | \mathcal{F}_j^*\right] = \mathbf{E}\left[a_j X_f(s) I(\tau(j) = j) A_{j,n}(s)\right] \tag{24}$$

with

$$\begin{aligned} A_{j,n}(s) &:= \mathbf{E}\left[\frac{e^{-\hat{S}_{n-j}}}{\prod_{i=1}^2 \left(\alpha_i \hat{a}_{n-j} + b_j + a_j \hat{b}_{n-j}\right)}; \hat{L}_{n-j} \geq 0 | \mathcal{F}_j^*\right] \\ &= \mathbf{E}\left[e^{-\hat{S}_{n-j}} \prod_{i=1}^2 \phi\left(\alpha_i, b_j, a_j; \hat{b}_{[n-j/2]}, \hat{V}_{[(n-j)/2]}, \hat{S}_{n-j}\right); \hat{L}_{n-j} \geq 0 | \mathcal{F}_j^*\right] \end{aligned}$$

and where we have taken the agreement that a random variable $\hat{\zeta} = \hat{\zeta}\left(\hat{Q}_1, .., \hat{Q}_n\right)$ has the same definition as $\zeta = \zeta(Q_1, ..., Q_n)$ but is generated by a sequence $\hat{Q}_1, .., \hat{Q}_n$ which is independent of $\mathcal{F}_j^*$ and has the same distribution as $Q_1, ..., Q_n$. By (23) and asymptotic representation (6) we conclude that for each j there exists a constant d_j such that

$$n^{3/2} A_{j,n}(s) \leq n^{3/2} \mathbf{E}\left[e^{-\hat{S}_{n-j}}; \hat{L}_{n-j} \geq 0\right] \leq d_j$$

for all n. Now, using Lemma 9 and the dominated convergence theorem we see that for each fixed j

$$\begin{aligned} \mathcal{A}_j(s) &:= \lim_{n\to\infty} n^{3/2} \mathbf{E}\left[\Delta_n(s); \tau(n) = j\right] \\ &= \mathbf{E}\left[a_j X_f(s) I(\tau(j) = j) \lim_{n\to\infty} n^{3/2} A_{j,n}(s)\right] \\ &= K_1 \mathbf{E}\left[a_j I(\tau(j) = j) X_f(s) A_j(s)\right], \end{aligned} \tag{25}$$

where

$$\begin{aligned} A_j(s) &:= \iiint \Phi\left(a_j, \frac{a_j}{1 - sf(0)}, b_j, b_j, a_j, a_j; u, v, -z\right) \times \\ &\times \mathbf{P}^+\left(B^+ \in du\right) \mathbf{P}_z^-\left(B^- \in dv\right) \nu_1(dz). \end{aligned} \tag{26}$$

To evaluate $\mathbf{E}\left[\Delta_n(s); \tau(n) = n - j\right]$ for a fixed j let $\hat{\mathcal{F}}_j$ be the the σ-algebra generated by a sequence of random laws $Q, \hat{Q}_1, .., \hat{Q}_j$, where $\hat{Q}_1, .., \hat{Q}_j$ are distributed as $Q_1, ..., Q_j$ and are independent of $Q_1, ..., Q_n$. Introduce a temporary notation

$$\hat{\alpha}_1 = \frac{1}{1 - \hat{f}_{0,j}(0)} \geq 1, \quad \hat{\alpha}_2 = \frac{1}{1 - \hat{f}_{0,j}(sf(0))} \geq 1. \tag{27}$$

By (14) we see that

$$\begin{aligned}
&\mathbf{E}[\Delta_n\left(s\right);\tau(n)=n-j] \\
= \ &\mathbf{E}\left[G_{n-j}(f_{n-j,n}\left(0\right))G_{n-j}(f_{n-j,n}\left(sf(0)\right)e^{-S_{n-j}}e^{-(S_n-S_{n-j})}X_f(s);\tau(n)=n-j\right] \\
&\quad = \mathbf{E}\left[\hat{a}_j I\left(\hat{L}_j\geq 0\right)X_f(s)B_{j,n}(s)\right],
\end{aligned}$$

where

$$B_{j,n}(s):=\mathbf{E}\left[\frac{e^{S_{n-j}}}{\prod_{i=1}^{2}\left(\hat{\alpha}_i+e^{S_{n-j}}b_{n-j}\right)};\tau(n-j)=n-j|\hat{\mathcal{F}}_j\right].$$

In view of (27) $\prod_{i=1}^{2}\left(\hat{\alpha}_i+e^{S_{n-j}}b_{n-j}\right)\geq 1$ which, along with (6), implies that for each j there exists a constant d_j' such that

$$n^{3/2}B_{j,n}(s)\leq n^{3/2}\mathbf{E}\left[e^{S_{n-j}};\tau(n-j)=n-j\right]\leq d_j'$$

for all n. Now Lemma 10, the duality principle for random walks and the dominated convergence theorem yield for each j

$$\begin{aligned}
\mathcal{B}_j(s) \quad : \quad &= \lim_{n\to\infty} n^{3/2}\mathbf{E}\left[\Delta_n\left(s\right);\tau(n)=n-j\right] \\
= \quad &\mathbf{E}\left[\hat{a}_j I\left(\hat{L}_j\geq 0\right)X_f(s)\lim_{n\to\infty}n^{3/2}B_{j,n}(s)\right] \\
&\quad = K_2\mathbf{E}\left[\hat{a}_j I\left(\hat{L}_j\geq 0\right)X_f(s)B_j(s)\right],
\end{aligned}$$

where (recall (18) and (19))

$$B_j(s):=\iiint\Psi\left(\hat{\alpha}_1,\hat{\alpha}_2;u,v,-z\right)\mathbf{P}_z^+\left(B^+\in du\right)\mathbf{P}^-\left(B^-\in dv\right)\mu_1\left(dz\right)$$

with $\hat{\alpha}_1$ and $\hat{\alpha}_2$ specified by (27). As a result letting first $n\to\infty$ and than $N\to\infty$ we get

$$H(s):=\lim_{n\to\infty}n^{3/2}\mathbf{E}\left[s^{Z_n};T=n+1\right]=\sum_{j=0}^{\infty}\left(\mathcal{A}_j(s)+\mathcal{B}_j(s)\right).$$

In particular,

$$\lim_{n\to\infty}n^{3/2}\mathbf{P}\left(T=n+1\right)=H(1). \tag{28}$$

Hence we conclude that

$$\lim_{n\to\infty}\mathbf{E}\left[s^{Z_n}|T=n+1\right]=\frac{H(s)}{H(1)}=:\mathbf{E}\left[s^Y\right]$$

and, by the dominated convergence theorem and continuity of the functions involved, $\lim_{s\uparrow 1}H(s)=H(1)$ showing that the limiting distribution has no atom at infinity.

The theorem is proved.

4 Functional limit theorem

The proof of Theorem 2 uses the same ideas as the proof of Theorem 1.

Let $l : \mathbb{R} \to [0,1]$ be the function specified by the equality

$$l(y) := yI\left(0 \leq y \leq 1\right) + I\left(y > 1\right).$$

For parameters $\alpha > 0, \beta > 0, \lambda > 0$, a three-dimensional vector $\mathbf{u} = (u_1, u_2, u_3)$ and variables v and x introduce the function

$$\theta\left(\alpha, \beta, \lambda; \mathbf{u}, v, x\right) := \frac{l(v)}{e^{-x}\alpha + \left(\lambda\alpha e^{-x} l(u_1) + l(u_2) l\left(v\right)\right)\left(\beta + \alpha\kappa(u_3)\right)}.$$

It is not difficult to check that if $u_2 \geq \varepsilon$ for some $\varepsilon > 0$ then $\phi\left(\alpha, \beta, \lambda; \mathbf{u}, v, x\right) \leq \beta^{-1}\varepsilon^{-1}$ and is continuous in $\mathbf{u}, v$ and x in the mentioned domain. For the particular case $\mathbf{u} = (1, 1, u)$ we use one more notation

$$\theta^*\left(\alpha, \beta, \lambda; u, v, x\right) := \frac{l(v)}{\alpha e^{-x} + \left(\lambda\alpha e^{-x} + l(v)\right)\left(\beta + \alpha\kappa(u)\right)}. \tag{29}$$

With the functions above and $\mathbf{v} = (v_1, v_2)$ we associate two more functions

$$\begin{aligned}
\Theta\left(\alpha, \beta, \lambda; \mathbf{u}, \mathbf{v}, x\right) &:= \prod_{i=1}^{2} \theta\left(\alpha, \beta, \lambda; \mathbf{u}, v_i, x\right), \\
\Theta^*\left(\alpha, \beta, \lambda; u, \mathbf{v}, x\right) &:= \prod_{i=1}^{2} \theta^*\left(\alpha, \beta, \lambda; u, v_i, x\right).
\end{aligned}$$

For fixed $t \in (0, 1)$ and $r \in [0, 1)$ introduce a random vector

$$\mathbf{U}_{[nt]} = \left(U_{[nt]}^{(1)}, U_{[nt]}^{(2)}, U_{[nt]}^{(3)}\right)$$

and a random variable $\tilde{V}_{[nt]}(r)$ by the equalities

$$U_{[nt]}^{(1)} := \frac{1 - \exp\left\{-\lambda\alpha e^{-S_{[nt]}}\right\}}{\lambda\alpha e^{-S_{[nt]}}},\ U_{[nt]}^{(2)} := \exp\left\{-\lambda\alpha e^{-S_{[nt]}}\right\}, U_{[nt]}^{(3)} := b_{[nt]}, \tag{30}$$

and

$$\tilde{V}_{[nt]}(r) := \left(1 - f_{[nt],n}(r)\right) e^{S_{[nt]} - S_n}. \tag{31}$$

Let, further,

$$V_k(r) := \left(1 - f_{k,0}(r)\right) e^{-S_k}.$$

It follows from Lemma 8 that

$$\lim_{k \to \infty} S_k = \infty \quad \mathbf{P}_x^+ \text{ - a.s. for any } x \geq 0.$$

This and Lemma 8 show that, as $n \to \infty$

$$\mathbf{U}_n \to \mathbf{U}_\infty := \left(1, 1, B^+\right) \quad \mathbf{P}_x^+ \text{ - a.s.} \tag{32}$$

Besides, if $\tilde{\mathbf{V}}_{[nt]}(r_1,r_2) := \left(\tilde{V}_{[nt]}(r_1), \tilde{V}_{[nt]}(r_2)\right)$ then as $k \to \infty$ (see, for instance, formula (3.1) in [6])

$$\begin{aligned} \mathbf{V}_k(r_1,r_2) &:= (V_k(r_1), V_k(r_2)) \\ &\to (V_\infty(r_1), V_\infty(r_2)) =: \mathbf{V}_\infty(r_1,r_2) \quad \mathbf{P}_x^- \text{ - a.s.,} \end{aligned} \tag{33}$$

where

$$1/V_\infty(r) := B^- + (1-r)^{-1}. \tag{34}$$

Observe that

$$\mathbf{P}^+\left(0 < B^+ < \infty\right) = \mathbf{P}^-\left(0 < V_\infty(r) < 1\right) = 1$$

by Lemma 8.

The arguments above combined with Lemma 4 imply the following statement.

Lemma 11 *Under conditions A1 to A4*

$$\begin{aligned} &\frac{\mathbf{E}\left[\Theta\left(\alpha,\beta,\lambda;\mathbf{U}_{[nt]},\tilde{\mathbf{V}}_{[nt]}(r_1,r_2),S_n\right)e^{-S_n};L_n \ge 0\right]}{\mathbf{E}\left[e^{-S_n};L_n \ge 0\right]} \\ \to\ &\iiiint \Theta^*\left(\alpha,\beta,\lambda;u,\mathbf{v},-z\right)\mathbf{P}^+\left(B^+ \in du\right)\mathbf{P}_z^-\left(\mathbf{V}_\infty(r_1,r_2)\in d\mathbf{v}\right)\nu_1\left(dz\right). \end{aligned}$$

Now we consider the function

$$\omega\left(\lambda;\mathbf{u},v,x\right) := \frac{l(v)}{1+\left(\lambda l(u_1)+l(u_2)l(v)e^x\right)\kappa(u_3)}.$$

Clearly, $\omega\left(\alpha;\mathbf{u},v,x\right)$ is continuous and does not exceed 1. For the particular case $\mathbf{u}=(1,1,u)$ we use one more notation

$$\omega^*\left(\lambda;u,v,x\right) := \frac{l(v)}{1+\left(\lambda+l(v)e^x\right)\kappa(u)}. \tag{35}$$

With the functions above and $\mathbf{v}=(v_1,v_2)$ we associate the functions

$$\Omega\left(\lambda;\mathbf{u},\mathbf{v},x\right) := \prod_{i=1}^{2}\omega\left(\lambda;\mathbf{u},v_i,x\right), \qquad \Omega^*\left(\lambda;u,\mathbf{v},x\right) := \prod_{i=1}^{2}\omega^*\left(\lambda;u,v_i,x\right).$$

Let now $\mathbf{U}_{[nt]} = \left(\tilde{U}_{[nt]}^{(1)}, \tilde{U}_{[nt]}^{(2)}, \tilde{U}_{[nt]}^{(3)}\right)$ and $\tilde{\mathbf{V}}_{[nt]}(r_1,r_2)$ be the same as in Lemma 12 above with one exception: one should take $\alpha=1$ in the definition of the components of $\mathbf{U}_{[nt]}$.

By Lemma 5 and relations (32) and (33) we see that the following statement is valid.

Lemma 12 *Under conditions A1 to A4*

$$\frac{\mathbf{E}\left[\Omega\left(\lambda;\mathbf{U}_{[nt]},\tilde{\mathbf{V}}_{[nt]}(r_1,r_2),S_n\right)e^{S_n};\tau(n)=n\right]}{\mathbf{E}\left[e^{S_n};\tau(n)=n\right]}$$
$$\to \iiiint \Omega^*\left(\lambda;u,\mathbf{v},-z\right)\mathbf{P}_z^+\left(B^+\in du\right)\mathbf{P}^-\left(\mathbf{V}_\infty(r_1,r_2)\in d\mathbf{v}\right)\mu_1\left(dz\right).$$

To go further, observe that if the offspring probability functions are fractional-linear, then by Lemma 7 for any $0\leq m<n, s\in[0,1]$, and $0\leq r_2<r_1<1$

$$\begin{aligned}&f_{0,m}(sf_{m,n}(r_1))-f_{0,m}(sf_{m,n}(r_2))\\&\qquad:=sG_{m,n}(s;r_1)G_{m,n}(s;r_2)e^{-S_n}\frac{r_1-r_2}{(1-r_1)(1-r_2)},\end{aligned}\tag{36}$$

where

$$\begin{aligned}G_{m,n}(s;r) &:= \frac{1-f_{m,n}(r)}{e^{-S_m}+(1-sf_{m,n}(r))\,b_m}\\ &= \frac{(1-f_{m,n}(r))\,e^{S_m-S_n}\times e^{S_n}}{1+\left[(1-s)\,e^{S_m}+s\,(1-f_{m,n}(r))\,e^{S_m-S_n}\times e^{S_n}\right]b_m} &(37)\\ &= \frac{(1-f_{m,n}(r))\,e^{S_m-S_n}}{e^{-S_n}+\left[(1-s)\,e^{S_m}e^{-S_n}+s\,(1-f_{m,n}(r))\,e^{S_m-S_n}\right]b_m} &(38)\\ &\leq G_{m,n}(1;r)=1-f_{0,n}(r)\leq 1. &(39)\end{aligned}$$

Introduce the notation

$$Y_t^{(n)}:=Z_{[nt]}e^{-S_{[nt]}},\ t\in(0,1).$$

Clearly, for $\lambda\geq 0$

$$\mathbf{E}\left[e^{-\lambda Y_t^{(n)}};T=n+1\right]=\mathbf{E}\left[e^{-\lambda Y_t^{(n)}}\left(f_{[nt],n+1}^{Z_{[nt]}}(0)-f_{[nt],n}^{Z_{[nt]}}(0)\right)\right]=\mathbf{E}[F_{n,t}(\lambda)],$$

where, in view of (36) and with $s=s(\lambda):=\exp\left\{-\lambda e^{-S_{[nt]}}\right\}$

$$\begin{aligned}F_{n,t}(\lambda) &:= f_{0,[nt]}\left(sf_{[nt],n}(f(0))\right)-f_{0,[nt]}\left(sf_{[nt],n}(0)\right)\\ &= sG_{[nt],n}(s;f(0))G_{[nt]n}(s;0)e^{-S_n}X_f(1).\end{aligned}$$

Lemma 13 *For each fixed j*

$$\mathcal{A}_j^*(\lambda):=\lim_{n\to\infty}n^{3/2}\mathbf{E}\left[F_{n,t}(\lambda);\tau(n)=j\right]=K_1\mathbf{E}\left[e^{-S_j}I\left(\tau(j)=j\right)X_f(1)\,A_j^*(\lambda)\right],$$

where

$$A_j^*(\lambda):=\iiiint\Theta^*\left(a_j,b_j,\lambda;u,\mathbf{v},-z\right)\mathbf{P}^+\left(B^+\in du\right)\mathbf{P}_z^-\left(\mathbf{V}_\infty(f(0),0)\in d\mathbf{v}\right)\nu_1\left(dz\right).$$

Proof. Let, as earlier, $\mathcal{F}_j^*$ be the σ-algebra generated by $Q_1, .., Q_j$ and Q. Denote

$$U_{j,n,t}^{(1)} := \frac{1-\exp\left\{-\lambda a_j e^{-\hat{S}_{[nt]-j}}\right\}}{\lambda a_j e^{-S_{[nt]}}}, U_{j,n,t}^{(2)} := \exp\left\{-\lambda a_j e^{-\hat{S}_{[nt]-j}}\right\}, U_{j,n,t}^{(3)} := \hat{b}_{[nt]-j},$$

$$\tilde{V}_{j,n,t}(r) := \left(1-\hat{f}_{[nt]-j,n-j}(r)\right) e^{\hat{S}_{[nt]-j}-\hat{S}_{n-j}}.$$

By splitting S_n as $(S_n - S_j) + S_j$ we deduce from (38) and (39) that for $s = s(\lambda) = \exp\left\{-\lambda a_j e^{-\hat{S}_{[nt]-j}}\right\}$ and $r \in [0,1)$

$$\begin{aligned}
&G_{nt,n}(s;r) I\left(\tau(n)=j\right) \\
&\quad \stackrel{d}{=} \frac{\tilde{V}_{j,n,t}(r)}{a_j e^{-\hat{S}_{n-j}} + \left[\lambda a_j U_{j,n,t}^{(1)} e^{-\hat{S}_{n-j}} + U_{j,n,t}^{(2)} \tilde{V}_{j,n,t}(r)\right]\left(b_j + a_j U_{j,n,t}^{(3)}\right)} \times \\
&\qquad\qquad \times I\left(\tau(j)=j\right) I\left(\hat{L}_{n-j} \geq 0\right) \\
&\quad = \theta\left(a_j, b_j, \lambda; \mathbf{U}_{j,n,t}, \tilde{V}_{j,n,t}(r), \hat{S}_{n-j}\right) I\left(\tau(j)=j\right) I\left(\hat{L}_{n-j} \geq 0\right) \leq 1.
\end{aligned}$$

Hence it follows that

$$\mathbf{E}\left[F_{n,t}(\lambda); \tau(n)=j\right] = \mathbf{E}\left[e^{-S_j} I\left(\tau(j)=j\right) X_f(1) A_{j,nt,n}^*(\lambda)\right],$$

where

$$A_{j,nt,n}^*(\lambda) := \mathbf{E}\left[\Theta\left(a_j, b_j, \lambda; \mathbf{U}_{j,n,t}, \tilde{\mathbf{V}}_{j,n,t}(f(0),0), \hat{S}_{n-j}\right) e^{-\hat{S}_{n-j}}; \hat{L}_{n-j} \geq 0 | \mathcal{F}_j^*\right].$$

Using now Lemma 13, the asymptotic representation (6) and applying the dominated convergence theorem we see that

$$\begin{aligned}
&\lim_{n\to\infty} n^{3/2} \mathbf{E}[F_{n,t}(\lambda); \tau(n)=j] \\
&\qquad = \mathbf{E}\left[e^{-S_j} I\left(\tau(j)=j\right) X_f(1) \lim_{n\to\infty} n^{3/2} A_{j,nt,n}^*(\lambda)\right] \\
&\qquad = K_1 \mathbf{E}\left[e^{-S_j} I\left(\tau(j)=j\right) X_f(1) A_j^*(\lambda)\right] = \mathcal{A}_j^*(\lambda),
\end{aligned}$$

as desired.

Lemma 14 *For any fixed* j

$$\mathcal{B}_j^*(\lambda) := \lim_{n\to\infty} n^{3/2} \mathbf{E}\left[F_{n,t}(\lambda); \tau(n)=n-j\right] = K_2 \mathbf{E}\left[e^{\hat{S}_j} I\left(\hat{L}_j \geq 0\right) X_f(1) B_j^*(\lambda)\right],$$

where

$$B_j^*(\lambda) := \iiint \Omega^*\left(\lambda; u, \mathbf{v}, -z\right) \mathbf{P}_z^+\left(B^+ \in du\right) \mathbf{P}^-\left(\mathbf{V}_\infty(\hat{f}_{0,j}(f(0)), \hat{f}_{0,j}(0)) \in d\mathbf{v}\right) \mu_1(dz).$$

Proof. Let $\mathcal{F}_{n-j+1,n+1}$ be the σ-algebra generated by $Q_{n-j+1},.., Q_{n+1}$. Denote

$$U_{n,t}^{(1)} := \frac{1-\exp\left\{-\lambda e^{-S_{nt}}\right\}}{\lambda e^{-S_{nt}}}, U_{n,t}^{(2)} := \exp\left\{-\lambda e^{-S_{nt}}\right\}, U_{n,t}^{(3)} := b_{nt},$$

$$\tilde{V}_{j,n,t}(r) := \left(1-f_{nt,n-j}(r)\right) e^{S_{nt}-S_{n-j}}.$$

By splitting S_n as $(S_n - S_j) + S_j$ and letting $s = s(\lambda) = \exp\left\{-\lambda e^{-S_{nt}}\right\}$ and $r \in [0,1)$ we deduce from (37) and (39) that

$$\begin{aligned} & G_{nt,n}(s;r) I\left(\tau(n) = n-j\right) = G_{nt,n-j}(s; f_{n-j,n}(r)) I\left(\tau(n) = n-j\right) \\ & \stackrel{d}{=} \frac{\tilde{V}_{j,n,t}(\hat{f}_{0,j}(r))}{1+\left[\lambda U_{n,t}^{(1)} + U_{n,t}^{(2)} \tilde{V}_{j,n,t}(\hat{f}_{0,j}(r)) e^{S_{n-j}}\right] U_{n,t}^{(3)}} I\left(\tau(n-j) = n-j\right) I\left(\hat{L}_j \geq 0\right) \\ = \quad & \omega\left(\lambda; \mathbf{U}_{n,t}, \tilde{V}_{j,n,t}(\hat{f}_{0,j}(r)), S_{n-j}\right) I\left(\tau(n-j) = n-j\right) I\left(\hat{L}_j \geq 0\right) \leq 1. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E}\left[F_{n,t}(\lambda); \tau(n) = n-j\right] &= \mathbf{E}[\mathbf{E}\left[F_{n,t}(\lambda) I\left(\tau(n) = n-j\right) | \mathcal{F}_{n-j+1,n+1}\right]] \\ &= \mathbf{E}\left[s e^{\hat{S}_j} X_f(1) I\left(\hat{L}_j \geq 0\right) B_{n,t,j}^{*}(\lambda)\right], \end{aligned}$$

where

$$B_{n,t,j}^{*}(\lambda) := \mathbf{E}\left[\Omega(\lambda; \mathbf{U}_{n,t}, \tilde{\mathbf{V}}_{j,n,t}(\hat{f}_{0,j}(f(0)), \hat{f}_{0,j}(0)), S_{n-j}) e^{S_{n-j}}; \tau(n-j) = n-j\right].$$

The needed statement follows now from Lemma 12 and the dominated convergence theorem.

The following lemma is crucial for our subsequent arguments.

Lemma 15 *For any $t \in (0,1)$, as $n \to \infty$*

$$\mathcal{L}\left(Y_t^{(n)} | T = n+1\right) \to \mathcal{L}(W)$$

weakly, where W is an a.s. positive proper random variable.

Proof. Relation (22) gives for sufficiently large n and all $N \geq N(\varepsilon)$

$$\begin{aligned} & \mathbf{E}[F_{n,t}(\lambda); N < \tau(n) < n-N] \\ & \qquad \leq \mathbf{P}(T = n+1; N < \tau(n) < n-N) \leq \varepsilon n^{-3/2}. \end{aligned} \tag{40}$$

By Lemmas 13 and 14 for each fixed N

$$\lim_{n\to\infty} n^{3/2} \mathbf{E}\left[F_{n,t}(\lambda); \tau(n) \notin [N+1, n-N]\right] = \sum_{j=0}^{N} \left(\mathcal{A}_j^{*}(\lambda) + \mathcal{B}_j^{*}(\lambda)\right)$$

which, in view of (28), implies

$$H^*(\lambda) = \mathbf{E}e^{-\lambda W} := \lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda Y_t^{(n)}}|T = n+1\right] = \frac{1}{H(1)}\sum_{j=0}^{\infty}\left(\mathcal{A}_j^*(\lambda) + \mathcal{B}_j^*(\lambda)\right).$$

It follows from the definitions (29) and (35) that

$$\lim_{\lambda\to\infty} \Theta^*\left(\alpha, \beta, \lambda; u, \mathbf{v}, x\right) = 0 \text{ and } \lim_{\lambda\to\infty} \Omega^*\left(\lambda; u, \mathbf{v}, x\right) = 0$$

leading by the dominated convergence theorem to $\lim_{\lambda\to\infty} H^*(\lambda) = 0$. Thus, the distribution of W has no atom at zero. On the other hand, again by the dominated convergence theorem,

$$\lim_{\lambda\downarrow 0} \mathcal{A}_j^*(\lambda) = K_1 \mathbf{E}\left[e^{-S_j} I\left(\tau(j) = j\right) X_f\left(1\right) A_j^*(0)\right],$$

where

$$A_j^*(0) := \iiiint \Theta^*\left(a_j, b_j, 0; u, \mathbf{v}, -z\right) \mathbf{P}^+\left(B^+ \in du\right) \mathbf{P}_z^-\left(\mathbf{V}_\infty(f(0), 0) \in d\mathbf{v}\right) \nu_1\left(dz\right).$$

We know by (34) that

$$\mathbf{V}_\infty(f(0), 0) = \left(\frac{1}{B^- + (1 - f(0))^{-1}}, \frac{1}{B^- + 1}\right)$$

which leads to

$$\begin{aligned} A_j^*(0) \quad &= \quad \iiint \Theta^*\left(a_j, b_j, 0; u, \frac{1}{v + (1 - f(0))^{-1}}, \frac{1}{v+1}, -z\right) \times \\ &\qquad \times \mathbf{P}^+\left(B^+ \in du\right) \mathbf{P}_z^-\left(B^- \in dv\right) \nu_1\left(dz\right). \end{aligned}$$

Recalling (15) and (29) we see that for any $h > 0$

$$\begin{aligned} \phi\left(ha_j, b_j, a_j; u, v, -z\right) \quad &= \quad \frac{1}{a_j h e^z + b_j + a_j u + a_j e^z v} \\ &= \quad \frac{(v+h)^{-1}}{a_j e^z + (v+h)^{-1}(b_j + a_j u)} \\ &= \quad \theta^*\left(a_j, b_j, 0; (v+h)^{-1}, -z\right). \end{aligned}$$

Hence, letting $h = 1$ and $h = (1 - f(0))^{-1}$ we obtain

$$\Theta^*\left(a_j, b_j, 0; u, \frac{1}{v + (1 - f(0))^{-1}}, \frac{1}{v+1}, -z\right) = \Phi\left(a_j, \frac{a_j}{1 - f(0)}, b_j, b_j, a_j, a_j; u, v, -z\right).$$

This equality, and representations (25) and (26) show that $\lim_{\lambda\downarrow 0} \mathcal{A}_j^*(\lambda) = \mathcal{A}_j(0)$ for any $j = 0, 1, ...$ In a similar way one can prove that $\lim_{\lambda\downarrow 0} \mathcal{B}_j^*(\lambda) = \mathcal{B}_j(0)$

for any $j = 0, 1, ...$ This implies $\lim_{\lambda \downarrow 0} H^*(\lambda) = H(1)$. Thus, the distribution of W has no atom at infinity.

The lemma is proved.

Proof of Theorem 2 . For fixed $\delta \in (0, 1/2)$ we introduce the process with constant paths

$$W_t^n := e^{-S_{[n\delta]}} Z_{[n\delta]}, t \in [\delta, 1-\delta] .$$

By Lemma 15 $\mathcal{L}(W_t^n, \delta \le t \le 1-\delta) \to \mathcal{L}(W)$ in distribution in the space $D[\delta, 1-\delta]$ with respect to the Skorokhod topology. Since the limiting process is continuous, we have convergence in the metric of uniform convergence as well. To prove the theorem it is sufficient to show that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[\delta,1-\delta]} |Y_t^n - W_t^n| > \varepsilon \middle| T = n+1 \right) = 0. \tag{41}$$

To simplify the subsequent arguments we introduce for $\delta \in (0, 1/2)$ and $M > 1$ the events

$$\mathcal{D}(\varepsilon, \delta) := \left\{ \sup_{t\in[\delta,1-\delta]} |Y_t^n - W_t^n| > \varepsilon \right\}, \quad \mathcal{K}(M, \delta) := \left\{ Y_{1-\delta}^n \in [M^{-1}, M] \right\}$$

and denote by $\mathcal{K}^c(M, \delta)$ the event complimentary to $\mathcal{K}(M, \delta)$. In view of Lemma 15 for any $\gamma > 0$ there exists M such that for all $n \ge n(\gamma, M)$

$$\mathbf{P}(\mathcal{D}(\varepsilon, \delta) \cap \mathcal{K}^c(M, \delta), T = n+1) \le \mathbf{P}(\mathcal{K}^c(M, \delta), T = n+1) \le \gamma n^{-3/2}.$$

Besides, by the arguments used to demonstrate (22) one can show that for any $\gamma > 0$ there exists N such that for all $n \ge n(\gamma, N)$

$$\begin{aligned} &\mathbf{P}(\mathcal{D}(\varepsilon, \delta) \cap \mathcal{K}(M, \delta), T = n+1; \tau(n) \in [N+1, n-N-1]) \\ &\qquad \le \mathbf{P}(T = n+1; \tau(n) \in [N+1, n-N-1]) \le \gamma n^{-3/2}. \end{aligned}$$

Clearly, for any $m < n$

$$\begin{aligned} R(Z_m, n) := \quad & f_{m,n+1}^{Z_m}(0) - f_{m,n}^{Z_m}(0) \le Z_m f_{m,n+1}^{Z_m - 1}(0) \left(f_{m,n+1}(0) - f_{m,n}(0) \right) \\ = \quad & Z_m f_{m,n+1}^{Z_m-1}(0) \left(1 - f_{m,n+1}(0) \right) \left(1 - f_{m,n}(0) \right) e^{S_m - S_n} X_{f_{n+1}}(1) . \end{aligned}$$

Hence, by the inequalities $1 - f_{m,n+1}(0) \le 1 - f_{m,n}(0) \le e^{S_n - S_m}$ and (21) and Assumption A1 we see that

$$R(Z_m, n) \le \frac{Z_m}{e^{S_m}} e^{S_n} X_{f_{n+1}}(1) \le \rho \frac{Z_m}{e^{S_m}} e^{S_n}. \tag{42}$$

On the other hand, by the inequality $1 - x \le e^{-x}$, $x > 0$, we get

$$f_{m,n+1}^{Z_m-1}(0) = \left(1 - (1 - f_{m,n+1}(0))\right)^{Z_m} \frac{1}{f_{m,n+1}(0)} \le e^{-Z_m(1-f_{m,n+1}(0))} \frac{1}{f_{m,n+1}(0)}$$

which, in view of the estimates

$$f_{m,n+1}(0) \geq f_{m+1}(0) \geq \chi, \quad \frac{1-f_{m,n}(0)}{1-f_{m,n+1}(0)} \leq \frac{1}{1-f_{m+1}(0)} \leq \frac{1}{1-\chi} \leq 2, \tag{43}$$

gives

$$\begin{aligned} & R(Z_m, n) \\ \leq \quad & \frac{Z_m}{e^{S_m}} e^{2S_m} (1-f_{m,n+1}(0))^2 e^{-Z_m(1-f_{m,n+1}(0))} \frac{e^{-S_n}}{f_{m,n+1}(0)} \frac{1-f_{m,n}(0)}{1-f_{m,n+1}(0)} X_{f_{n+1}}(1) \\ & \leq 2\rho\chi^{-1} \frac{Z_m}{e^{S_m}} e^{-S_n} \sup_{x\geq 0} x^2 \exp\left\{-\frac{Z_m}{e^{S_m}} x\right\}. \end{aligned}$$

For $m = [n(1-\delta)]$ inequalities (42) and (43) give

$$R(Z_{[n(1-\delta)]}, n) I\left(\mathcal{K}(M,\delta)\right) \leq \rho M e^{S_n} \tag{44}$$

and

$$\begin{aligned} R(Z_{[n(1-\delta)]}, n) I\left(\mathcal{K}(M,\delta)\right) \quad & \leq \quad 2\rho\chi^{-1} M e^{-S_n} \sup_{x\geq 0} x^2 e^{-xM^{-1}} \\ & \leq \quad 8\rho\chi^{-1} M^3 e^{-2} e^{-S_n}. \end{aligned} \tag{45}$$

Recalling that by Lemma 3.8 in [2]

$$\mathbf{P}\left(\mathcal{D}\left(\varepsilon,\delta\right)|\Pi\right) \leq \left(\varepsilon^{-2}\left[\sum_{i=n\delta}^{[n(1-\delta)]} \eta_{i+1} e^{-S_i} + e^{-S_{[n(1-\delta)]}} - e^{-S_{[n\delta]}}\right]\right) \wedge 1 =: \mathcal{U}_n \tag{46}$$

we get by means of (45) for a fixed $j \in \mathbb{N}_0$

$$\begin{aligned} \Xi_{(j)}(n) := \quad & \mathbf{P}\left(\mathcal{D}\left(\varepsilon,\delta\right) \cap \mathcal{K}(M,\delta), T = n+1; \tau(n) = j\right) \\ = \quad & \mathbf{E}\left[I\left(\mathcal{D}\left(\varepsilon,\delta\right) \cap \mathcal{K}(M,\delta)\right)\left(f_{[n(1-\delta)],n+1}^{Z_{[n(1-\delta)]}}(0) - f_{[n(1-\delta)],n}^{Z_{[n(1-\delta)]}}(0)\right); \tau(n) = j\right] \\ \leq \quad & 8\rho\chi^{-1} M^3 e^{-2} \mathbf{E}\left[I\left(\mathcal{D}\left(\varepsilon,\delta\right)\right) e^{-S_n}; \tau(n) = j\right] \\ \leq \quad & 8\rho\chi^{-1} M^3 e^{-2} \mathbf{E}\left[\mathcal{U}_n e^{-S_n}; \tau(n) = j\right]. \end{aligned}$$

By Lemma 3.1 in [2], for all $x \geq 0$

$$\mathcal{U}_n \to 0 \qquad \mathbf{P}_x^+ - \text{a.s.}$$

Hence, applying the arguments similar to those used in the proof of Lemma 13 we see that, as $n \to \infty$, $\Xi_{(j)}(n) = o(n^{-3/2})$ for each fixed j. Further, using inequality (44) we get

$$\begin{aligned} \Xi^{(j)}(n) := \quad & \mathbf{P}\left(\mathcal{D}\left(\varepsilon,\delta\right) \cap \mathcal{K}(M,\delta), T = n+1; \tau(n) = n-j\right) \\ & \leq \rho M \mathbf{E}\left[\mathcal{U}_n e^{S_n}; \tau(n) = n-j\right]. \end{aligned}$$

Applying now arguments similar to those used to demonstrate Lemma 14 one can show that as $n \to \infty$, $\Xi^{(j)}(n) = o(n^{-3/2})$ for each fixed j.

Combining the estimates above and taking first the limit as $n \to \infty$ then as $\gamma \downarrow 0$ and, finally, as $M \to \infty$ we arrive at (41).

The theorem is proved.

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